Homework 7
MTH 829 Complex Analysis

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Lemma 0.1 (for Exercise IX.5.3)**.** *Let $z_0 \in \mathbb{C}$. For $t \in \mathbb{R}$, the following limit converges uniformly.*

$$\lim_{R \to \infty} \frac{Re^{it}}{Re^{it} - z_0} = 1$$

Proof. Let $\epsilon > 0$. Set $M = |z_0| \left(1 + \frac{1}{\epsilon}\right)$. For $t \in \mathbb{R}$, $|e^{it}| = 1$.

$$M < R \implies |z_0| \left(1 + \frac{1}{\epsilon}\right) < R \implies |z_0| + \frac{|z_0|}{\epsilon} < |Re^{it}| \implies \frac{|z_0|}{\epsilon} < |Re^{it}| - |z_0|$$

By the triangle inequality,

$$|Re^{it}| - |z_0| \leq |Re^{it} - z_0|$$

Thus

$$\frac{|z_0|}{\epsilon} \leq |Re^{it} - z_0| \implies |z_0| \leq |Re^{it} - z_0|\epsilon \implies \left| \frac{z_0}{Re^{it} - z_0} \right| \leq \epsilon$$

We can rewrite this final inequality as

$$\left| \frac{z_0}{Re^{it} - z_0} \right| = \left| \frac{Re^{it} - (Re^{it} - z_0)}{Re^{it} - z_0} \right| = \left| \frac{Re^{it}}{Re^{it} - z_0} - 1 \right| \leq \epsilon$$

and thus

$$\lim_{R \to \infty} \frac{Re^{it}}{Re^{it} - z_0} = 1$$

Since M does not depend on t, convergence is uniform. □

Proposition 0.2 (Exercise IX.5.3)**.** *Let $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\lim_{R \to \infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{1}{z - z_0} dz = \begin{cases} \frac{1}{2} & \operatorname{Im} z_0 > 0 \\ -\frac{1}{2} & \operatorname{Im} z_0 < 0 \end{cases}$$

Proof. First, assume $\operatorname{Im} z_0 > 0$. For $R > 0$ define γ_R to be the curve $\gamma_R(t) = Re^{it}$ for $t \in [0, \pi]$ and define Γ_R to be the closed curve $\Gamma_R = [-R, R] \cup \gamma_R$ so we have

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{1}{z - z_0} dz = \frac{1}{2\pi i} \lim_{R\to\infty} \left(\int_{\Gamma_R} \frac{1}{z - z_0} dz - \int_{\gamma_R} \frac{1}{z - z_0} dz \right)$$

For R sufficiently large, z_0 is is in the interior of Γ_R, and $\operatorname{ind}_{\Gamma_R}(z_0) = 1$. Thus

$$\int_{\Gamma_R} \frac{1}{z - z_0} dz = 2\pi i \implies \lim_{R\to\infty} \int_{\Gamma_R} \frac{1}{z - z_0} dz = 2\pi i$$

Now we compute the integral over γ_R.

$$\int_{\gamma_R} \frac{1}{z - z_0} dz = \int_0^{\pi} \frac{iRe^{it}}{Re^{it} - z_0} dt = i \int_0^{\pi} \frac{Re^{it}}{Re^{it} - z_0} dt$$

We want to take the limit as $R \to \infty$. Using the previous lemma and the result from VI.11 of Sarason, we can move the limit inside the integral, to get

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z - z_0} dz = i \lim_{R\to\infty} \int_0^{\pi} \frac{Re^{it}}{Re^{it} - z_0} dt = i \int_0^{\pi} \lim_{R\to\infty} \frac{Re^{it}}{Re^{it} - z_0} dt = i \int_0^{\pi} 1 dt = i\pi$$

Thus for $\operatorname{Im} z_0 > 0$, we get

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{1}{z - z_0} dz = \frac{1}{2\pi i}(2\pi i - \pi i) = \frac{1}{2}$$

Essentially the same argument will show the other equality in the case $\operatorname{Im} z_0 < 0$. Suppose $\operatorname{Im} z_0 < 0$, and now take γ_R to be the curve $\gamma_R(t) = Re^{-it}$ for $t \in [0, \pi]$, and set $\Gamma_R = \gamma_R \cup [-R, R]$. Then

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{1}{z - z_0} dz = \frac{1}{2\pi i} \lim_{R\to\infty} \left(\int_{\Gamma_R} \frac{1}{z - z_0} dz - \int_{\gamma_R} \frac{1}{z - z_0} dz \right)$$

For R sufficiently large, z_0 is in the interior of Γ_R, and the winding number of Γ_R around z_0 is -1, so

$$\int_{\Gamma_R} \frac{1}{z - z_0} dz = -2\pi i$$

Now we compute the integral around γ_R in this case. We could just conclude by symmetry that the integral will turn out to be the same as the previous integral over a semicircle, or we can just compute

$$\int_{\gamma_R} \frac{1}{z - z_0} dz = \int_0^{\pi} \frac{-iRe^{-it}}{-Re^{-it} - z_0} dt = i \int_0^{\pi} \frac{Re^{-it}}{Re^{-it} + z_0} dt$$

and taking the limit gives

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z - z_0} dz = i \lim_{R\to\infty} \int_0^{\pi} \frac{Re^{-it}}{Re^{-it} + z_0} dt = i \int_0^{\pi} \lim_{R\to\infty} \frac{Re^{-it}}{Re^{-it} + z_0} dt = i \int_0^{\pi} 1 dt = i\pi$$

Thus if $\operatorname{Im} z_0 < 0$, we get

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{1}{z - z_0} dz = \frac{1}{2\pi i}(-2\pi i - \pi i) = -\frac{1}{2}$$

□

Proposition 0.3 (Exercise IX.17.1)**.** *Let $G \subset \mathbb{C}$ be connected and open. If $z_1, z_2 \in G$, then there is a polygonal path from z_1 to z_2 that lies in G.*

Proof. Let G be as described and let $z \in G$. Let $U \subset G$ be the subset such that for every $w \in U$, there is a polygonal path from z to w. Since G is open, there is an open ball of some radius ϵ_w containing w that is contained in G,

$$w \subset B(w, \epsilon_w) \subset G$$

Then every point $\alpha \in B(w, \epsilon_w)$ can be connected to z via a polygonal path in G, since we can take a polygonal path from z to w (in G) and then adjoin a straight line path from w to α (which lies in G since $B(w, \epsilon_w) \subset G$). Thus $B(w, \epsilon_w) \subset U$. Thus, for any $w \in U$, there is an open ball containing w that is contained in U. Thus U is open.

Now suppose $y \in G \setminus U$, that is, there is no polygonal path from z to w lying in G. Since $y \in G$, there is an open ball containing y contained in G,

$$y \in B(y, \epsilon_y) \subset G$$

If a point $\beta \in B(y, \epsilon_y)$ could be connected to z via a polygonal path in G, then by using that path and a straight line path from β to y, we would have a polygonal path from y to z, which is a contradiction. Thus there is no polygonal path from β to z lying in G. Thus $B(y, \epsilon_y) \subset G / \setminus U$. Consequently, $G \setminus U$ is open, so U is closed.

Since U is both open, closed, and not empty ($z \in U$), and G is connected, U must be equal to G. Thus any two points in G can be connected by a polygonal path in G. □

Proposition 0.4 (Exercise 1)**.** *Let G be the infinite vertical strip $\{x + iy : -1 < x < 1\}$. Let $f : G \setminus \{0\} \to \mathbb{C}$ be holomorphic, so that $\lim_{z\to\infty} f(z)$ exists and is in $\mathbb{C}$. Then for $x \in (0, 1)$,*

$$\lim_{R\to\infty} \int_{-R}^{R} \big(f(x + iy)) - f(-x + iy)\big) dy = 2\pi \operatorname{res}_0 f$$

Proof. For $R > 0$ and $x \in (0, 1)$, define $\gamma_{R,x}$ to be the closed rectangular curve with vertices $\pm x \pm iR$, oriented counterclockwise. More concretely, $\gamma_{R,x}$ is the union of four line segments as below:

$$\gamma_{R,x} = [x - iR, x + iR] \cup [x + iR, -x + iR] \cup [-x + iR, -x - iR] \cup [-x - iR, x - iR]$$

By construction, the winding number of $\gamma_{R,x}$ around zero is one, so by the Residue Theorem,

$$\int_{\gamma_{R,x}} f(z) dz = 2\pi i \operatorname{res}_0 f \implies \lim_{R\to\infty} \int_{\gamma_{R,x}} f(z) dz = 2\pi i \operatorname{res}_0 f$$

First we show that the contributions from the integrals over the horizontal line segments cancel each other in the limit as $R \to \infty$. We can write them as

$$\int_{[x+iR,-x+iR]} f(z)dz = \int_{x}^{-x} f(t+iR)dt = -\int_{-x}^{x} f(t+iR)dt$$
$$\int_{[-x-iR,x-iR]} f(z)dz = \int_{-x}^{x} f(t-iR)dt$$

Then their sum is

$$-\int_{-x}^{x} f(t+iR)dt + \int_{x}^{x} f(t-iR)dt = \int_{-x}^{x} f(t-iR) - f(t+iR)dt$$

Let $L = \lim_{z\to\infty} f(z)$. Let $\epsilon > 0$. Then there exists $M > 0$ so that

$$|z| > M \implies |f(z) - L| < \epsilon$$

which implies

$$|z|, |w| > M \implies |f(z) - f(w)| < 2\epsilon$$

Thus there exists $M > 0$ so that

$$|f(t-iR) - f(t+iR)| < 2\epsilon$$

thus

$$\left|\int_{-x}^{x} f(t-iR) - f(t+iR)dt\right| \le \int_{-x}^{x} |f(t-iR) - f(t+iR)|dt \le \int_{-x}^{x} 2\epsilon dt = 4x\epsilon$$

Since $\epsilon > 0$ was arbitrary, this implies

$$\lim_{R\to\infty}\left|\int_{-x}^{x} f(t-iR) - f(t+iR)dt\right| = 0 \implies \lim_{R\to\infty}\int_{-x}^{x} f(t-iR) - f(t+iR)dt = 0$$

Now that we know that the horizontal parts of the rectangular integral don't contribute, we have

$$\lim_{R\to\infty}\left(\int_{[x-iR,x+iR]} f(z)dz + \int_{[-x+iR,-x-iR]} f(z)dz\right) = 2\pi i \operatorname{res}_0 f$$

We can rewrite these integrals as

$$\int_{[x-iR,x+iR]} f(z)dz = \int_{-R}^{R} if(x+iy)dy$$
$$\int_{[-x+iR,-x-iR]} f(z)dz = \int_{-R}^{R} -if(-x+iy)dy$$

Thus

$$\lim_{R\to\infty}\int_{-R}^{R} \big(if(x+iy) - if(-x+iy)\big)dy = 2\pi i \operatorname{res}_0 f$$

and cancelling out a factor of i,

$$\lim_{R\to\infty}\int_{-R}^{R} \big(f(x+iy) - f(-x+iy)\big)dy = 2\pi \operatorname{res}_0 f$$

□

Proposition 0.5 (Exercise 2). *Let G be the slit plane $\mathbb{C} \setminus (-\infty, 0]$. Let $f : G \to \mathbb{C}$ be holomorphic, so that*

$$\lim_{z \to \infty} f(z) = 0 \qquad \lim_{z \to 0} |z| f(z) = 0$$

and for all $x \in (-\infty, 0)$, we have locally uniform convergence of the following two limits.

$$\lim_{y \to 0^+} f(x + iy) \qquad \lim_{y \to 0^-} f(x + iy)$$

Define functions $\phi_+, \phi_- : (-\infty, 0) \to \mathbb{C}$ by

$$\phi_+(x) = \lim_{y \to 0^+} f(x + iy) \qquad \phi_-(x) = \lim_{y \to 0^-} f(x + iy)$$

Then for $z \in G$,

$$f(z) = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{-R}^{0} \frac{\phi_+(x) - \phi_-(x)}{x - z} dx$$

Proof. Let $z \in G$, and define

$$g(w) = \frac{f(w)}{w - z}$$

Then g is holomorphic on $G \setminus \{z\}$, and $\operatorname{res}_z g = f(z)$. Let $R > 0$. Define

$$R' = \sqrt{R^2 + \frac{1}{R^2}}$$

Define curves $\beta_R, \alpha_R, \eta_R, \chi_R$ by

$$\begin{aligned}
\beta_R(t) &= \frac{1}{R} e^{-it} && t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
\alpha_R(t) &= R' e^{it} && t \in \left[-\pi + \frac{1}{R}, \pi - \frac{1}{R}\right] \\
\eta_R(t) &= t + \frac{i}{R} && t \in [-R, 0] \\
\chi_R(t) &= -t - \frac{i}{R} && t \in [0, R]
\end{aligned}$$

Let Γ_R be the union of these four, so Γ_R is a closed curve. For sufficiently large R, z lies in the interior of Γ_R. For R large enough that z is in the interior of Γ_R, the winding number is $\operatorname{ind}_{\Gamma_R} z = 1$. Then by the Residue Theorem,

$$\int_{\Gamma_R} g(w) dw = 2\pi i \operatorname{ind}_{\Gamma_R}(z) \operatorname{res}_z g = 2\pi i f(z)$$

Since the RHS above is independent of R, taking the limit as $R \to \infty$ gives

$$\lim_{R \to \infty} \int_{\Gamma_R} g(w) dw = 2\pi i f(z)$$

Now we compute the integrals over each of the pieces individually ($\alpha_R, \beta_R, \eta_R, \chi_R$), starting with β_R. (Throughout, we assume that R is large enough that z lies on the "outside" of the circle we'd get by "completing" β_R. We can assume this because we only care about what happens as $R \to \infty$.) We can use the arc length estimate, we get

$$\left|\int_{\beta_R} g(w)dw\right| = \left|\int_{\beta_R} \frac{f(w)}{w-z}dw\right| \le \text{length}(\beta_R) \max_{w\in\beta_R}\left\{\frac{f(w)}{w-z}\right\}$$

For $w \in \beta_R$, $|w| = \frac{1}{R}$, and length$(\beta_R) = \frac{\pi}{R}$, so

$$\text{length}(\beta_R) \max_{w\in\beta_R}\left\{\frac{f(w)}{w-z}\right\} = \pi \max_{w\in\beta_R}\left\{\frac{|w|f(w)}{w-z}\right\}$$

As $R \to \infty$, for $w \in \beta_R$, we have $|w| \to 0$. By hypothesis, $\lim_{w\to 0} |w|f(w) = 0$, so

$$\lim_{R\to\infty} \max_{w\in\beta_R}\left\{\frac{|w|f(w)}{w-z}\right\} = 0$$

Combining this with the previous inequalities,

$$\lim_{R\to\infty}\int_{\beta_R} g(w)dw = 0$$

Now consider α_R. (We assume that R is large enough that z lies on the "inside" of the circle we'd get by "completing" α_R. We can assume this because we only care about what happens as $R \to \infty$.) Note that length$(\alpha_R) \le 2\pi R'$, so using the arc length inequality,

$$\left|\int_{\alpha_R} g(w)dw\right| \le \text{length}(\alpha_R) \max_{w\in\alpha_R}\left\{\frac{f(w)}{w-z}\right\} \le 2\pi R' \max_{w\in\alpha_R}\left\{\frac{f(w)}{w-z}\right\} = 2\pi \max_{w\in\alpha_R}\left\{\frac{R'f(w)}{w-z}\right\}$$

For $w \in \alpha_R$, as $R' \to \infty$, $|w-z|$ approaches $|w| = R'$, so

$$\lim_{R\to\infty} \max_{w\in\alpha_R}\left\{\frac{R'}{w-z}\right\} = 1$$

Thus

$$\lim_{R\to\infty} 2\pi \max_{w\in\alpha_R}\left\{\frac{R'f(w)}{w-z}\right\} = 2\pi \lim_{R\to\infty} \max_{w\in\alpha_R}\{f(w)\}$$

By hypothesis, $\lim_{w\to\infty} f(w) = 0$, so

$$\lim_{R\to\infty}\left|\int_{\alpha_R} g(w)dw\right| \le 2\pi \lim_{R\to\infty} \max_{w\in\alpha_R}\{f(w)\} = 0$$

thus

$$\lim_{R\to\infty}\int_{\alpha_R} g(w)dw = 0$$

Now we consider η_R.

$$\int_{\eta_R} g(w)dw = \int_{\eta_R} \frac{f(w)}{w-z}dw = \int_{-R}^{0} \frac{f(\eta_R(x))}{\eta_R(x)-z}dx = \int_{-R}^{0} \frac{f(x+i/R)}{(x+i/R)-z}dx$$

Since $\lim_{y \to 0^+} f(x+iy) = \phi_+(x)$ converges uniformly,

$$\lim_{R\to\infty} \int_{-R}^{0} \frac{f(x+i/R)}{(x+i/R)-z} dx = \lim_{R\to\infty} \int_{-R}^{0} \lim_{R\to\infty} \frac{f(x+i/R)}{(x+i/R)-z} dx = \lim_{R\to\infty} \int_{-R}^{0} \frac{\phi_+(x)}{x-z} dx$$

We can treat χ_R similarly. We get a negative sign from $\chi_R'(x)$, but then we replace $-x$ by x and change the integral from $[0,R]$ to $[0,-R]$, and then introduce another negative sign from changing the order of integration from $[0,-R]$ to $[-R,0]$, so the negative signs cancel out.

$$\begin{aligned}
\int_{\chi_R} g(w)dw &= \int_{\chi_R} \frac{f(w)}{w-z} dw = \int_0^R \frac{f(\chi_R(x))}{\chi_R(x)-z} \chi_R'(x) dx \\
&= -\int_0^R \frac{f(-x-i/R)}{(-x-i/R)-z} dx = -\int_0^{-R} \frac{f(x-i/R)}{(x-i/R)-z} dx \\
&= \int_{-R}^{0} \frac{f(x-i/R)}{(x-i/R)-z} dx
\end{aligned}$$

Since $\lim_{y \to 0^-} f(x+iy) = \phi_-(x)$ converges uniformly,

$$\lim_{R\to\infty} \int_{-R}^{0} \frac{f(x-i/R)}{(x-i/R)-z} dx = \lim_{R\to\infty} \int_{-R}^{0} \lim_{R\to\infty} \frac{f(x-i/R)}{(x-i/R)-z} dx = \lim_{R\to\infty} \int_{-R}^{0} \frac{\phi_-(x)}{x-z} dx$$

Summing up what we have shown so far,

$$\begin{aligned}
&\lim_{R\to\infty} \int_{\beta_R} g(w)dw = 0 \\
&\lim_{R\to\infty} \int_{\alpha_R} g(w)dw = 0 \\
&\lim_{R\to\infty} \int_{\eta_R} g(w)dw = \lim_{R\to\infty} \int_{-R}^{0} \frac{\phi_+(x)}{x-z} dx \\
&\lim_{R\to\infty} \int_{\chi_R} g(w)dw = -\lim_{R\to\infty} \int_{-R}^{0} \frac{\phi_-(x)}{x-z} dx
\end{aligned}$$

Using these equalities, we can can rewrite the integral over Γ_R to get

$$\begin{aligned}
2\pi i f(z) &= \lim_{R\to\infty} \int_{\Gamma_R} g(w)dw \\
&= \lim_{R\to\infty} \left(\int_{\beta_R} g(w)dw + \int_{\alpha_R} g(w)dw + \int_{\eta_R} g(w)dw + \int_{\chi_R} g(w)dw \right) \\
&= \lim_{R\to\infty} \int_{\eta_R} g(w)dw + \lim_{R\to\infty} \int_{\chi_R} g(w)dw \\
&= \lim_{R\to\infty} \int_{-R}^{0} \frac{\phi_+(x)}{x-z} dx - \lim_{R\to\infty} \int_{-R}^{0} \frac{\phi_-(x)}{x-z} dx \\
&= \lim_{R\to\infty} \int_{-R}^{0} \frac{\phi_+(x)-\phi_-(x)}{x-z} dx
\end{aligned}$$

Thus

$$f(z) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{0} \frac{\phi_+(x) - \phi_-(x)}{x - z} dx$$

$\square$

Proposition 0.6 (Exercise 3)**.** *Let* $\sqrt{\cdot}$ *denote the principal branch of the square root on* $\mathbb{C} \setminus (-\infty, 0]$. *Then for* $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$\frac{1}{\sqrt{z}} = \frac{1}{\pi} \int_0^\infty \frac{1}{(x+z)\sqrt{x}} dx$$

Proof. Define $f(z) = \frac{1}{\sqrt{z}}$. Then f is holomorphic on $\mathbb{C} \setminus (\infty, 0]$, and

$$\lim_{z\to\infty} |f(z)| = \lim_{z\to\infty} \frac{1}{|\sqrt{z}|} = 0 \implies \lim_{z\to\infty} f(z) = 0$$

and

$$\lim_{z\to 0} \big||z|f(z)\big| = \lim_{z\to 0} \frac{|z|}{|\sqrt{z}|} = \lim_{z\to 0} \frac{|z|}{|z|^{1/2}} = \lim_{z\to 0} |z|^{1/2} = 0 \implies \lim_{z\to 0} |z|f(z) = 0$$

Define $\phi_+(x) = \lim_{y\to 0^+} f(x+iy)$ and $\phi_-(x) = \lim_{y\to 0^-} f(x+iy)$. For $x \in (-\infty, 0)$, we have

$$\lim_{y\to 0^+} \sqrt{x+iy} = i\sqrt{-x}$$
$$\lim_{y\to 0^-} \sqrt{x+iy} = -i\sqrt{-x}$$

with locally uniform convergence, so

$$\phi_+(x) = \lim_{y\to 0^+} \frac{1}{\sqrt{x+iy}} = \frac{1}{i\sqrt{-x}}$$
$$\phi_-(x) = \lim_{y\to 0^-} \frac{1}{\sqrt{x+iy}} = \frac{1}{-i\sqrt{-x}}$$

Then using Exercise 2, for $z \in \mathbb{C} \setminus (-\infty, 0)$,

$$f(z) = \frac{1}{\sqrt{z}} = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{0} \frac{\phi_+(x) - \phi_-(x)}{x - z} dx$$

As a preliminary simplification, if $x \in (0, \infty)$, then

$$\phi_+(-x) - \phi_-(-x) = \frac{1}{i\sqrt{x}} - \frac{1}{-i\sqrt{x}} = \frac{2}{i\sqrt{x}} = \frac{-2i}{\sqrt{x}}$$

Now we rewrite the integral as

$$\frac{1}{\sqrt{z}} = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{0} \frac{\phi_+(x) - \phi_-(x)}{x - z} dx = \frac{1}{2\pi i} \lim_{R\to\infty} \int_0^R \frac{\phi_+(-x) - \phi_-(-x)}{-x - z} dx$$
$$= \frac{1}{2\pi i} \lim_{R\to\infty} \int_0^R \frac{\frac{-2i}{\sqrt{x}}}{-x - z} dx = \frac{1}{\pi} \lim_{R\to\infty} \int_0^R \frac{\frac{1}{\sqrt{x}}}{x + z} dx = \frac{1}{\pi} \int_0^\infty \frac{1}{(x+z)\sqrt{x}} dx$$

Thus

$$\frac{1}{\sqrt{z}} = \frac{1}{\pi} \int_0^\infty \frac{1}{(x+z)\sqrt{x}} dx$$

$\square$